International Journal of Theoretical Physics, Vol. 32, No. 11, 1993

Null Tetrad Formulation of Non-Abelian Dyons

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Received July 10, 1992

Starting with a brief description of dyons and gravito-dyons, combined field equations and the equation of motion for generalized electromagnetic and generalized gravito-Heavisidian fields are derived in a manifestly covariant way. A non-Abelian gauge theory of dyons and gravito-dyons is described in terms of a generalized Yang–Mills potential, field strengths, and generalized field equations each carrying electric and magnetic constituents. A null tetrad formulation of a generalized Yang–Mills potential and field strength of dyons is discussed in detail in terms of symmetric spinors and spin coefficients. Generalized Yang–Mills field equations of dyons are rewritten in terms of null tetrad notation, and dyon solutions of source-free Dirac equations are obtained.

1. INTRODUCTION

The theory of magnetic charge, propounded by Dirac (1931) for the symmetry of Maxwell's equations, in a manifest way was based on "classical magnetic monopole" without prediction of its mass. If the "classical electron radius" is equal to the "classical monopole radius," the mass of monopole becomes $M_m = 4700m_e \approx 2.4$ GeV (Craigie, 1986). Such monopoles are known as pointlike monopoles. 't Hooft (1974) and Polyakov (1974) examined the existence of monopoles in grand unified theories and estimated the monopole mass $M_m \approx M_x/\alpha_v$, where M_x is the scale of symmetry breaking leading to the $U(1)$ factor and α_v is the unification coupling constant. Such monopoles are known as superheavy monopoles, with their estimated mass $M_m \approx 10^{16}$ GeV. In supersymmetric grand unified theories the monopoles seem to be more massive and if gravity is taken into the unifying picture, then monopoles could be even more massive, $M_m \approx 10^{19}$ GeV (Craigie, 1986). Magnetic monopoles of

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0020-7748/93/1100-2099$07.00/0

lowest mass are expected to be stable since magnetic charge should be conserved like electric charges. Observations made by Cabrera (1982) also aroused the interest of many to study monopoles and their potential importance in connection with quark confinement ('t Hooft, 1976; Cho, 1980), the magnetic condensation of vacuum ('t Hooft, 1981), *CP* violation (Witten, 1979), proton decay (Rubakov, 1982; Callan, 1982), grand unified theories (Georgi and Glashow, 1974), and supersymmetry. Despite of the enormous potential importance of monopoles, the formalism necessary to describe them has been clumsy and manifestly noncovariant. For instance, in addition to the problem raised by the Dirac (1931) veto, there is Goldhaber's (1976) paradox about the spin-statistics relationship. This paradox could be solved only by taking electric and magnetic charges together (i.e., dyons). Schwinger (1969) give the existence of such dyons was pointed out by Julia and Zee (1975) by extending the 't Hooft–Polyakov model in the non-Abelian gauge theory of monopoles.

Quantum mechanical excitations of the fundamental monopoles include dyons. Dyons arise automatically from the semiclassical quantization of the global charge rotation degrees of freedom of monopoles. In connection with the explanation of *CP* violation in terms of nonzero vacuum angle (Witten, 1979) of the world. The monopoles are necessarily dyons and the Dirac quantization condition permits dyons to have an anomalous electric charge. The existence of these dyons can remove the objections (Goldhaber, 1976) to the spin-statistics relationship for monopoles and also those raised by pantaleone (1963) toward the experimental results of Fairbank *et al.* (1981). On the other hand, Carmeli (1977*a*,*b*, 1982; Carmeli and Huleihil, 1984) described a null tetrad formalism of the Yang–Mills field equations and derived the exact solutions of the Yang–Mills source-free equations, written in null tetrad notation. These solutions include both "electric" and "magnetic" charges and may also be considered as dyon solutions.

Cabibbo and Ferrari (1962) introduced the idea of two-four potentials for avoiding arbitrary string variables. Rajput *et al.* (1983*a*, 1984, 1986, 1988) constructed a manifestly covariant self-consistent quantum field theory of dyons each carrying generalized charge, generalized potential, generalized current, field, and generalized field strength antisymmetric tensor as complex quantities with electric and magnetic charges as their real and imaginary constituents. Rajput *et al.* (1983*b*, 1986; Rajput and Kumar, 1983) analyzed the behavior of dyonic fields in non-Abelian gauge theory and developed (Rajput *et al.*, 1982, 1985) quaternionic formulation of generalized fields of dyons. Cantani (1980) and Bisht *et al.* (1990) postulated the existence of dual mass (Heavisidian monopole) in a linear gravitational field, and keeping in view the discrepancies of the Dirac

(1931) veto, the structural symmetry between generalized electromagnetic and gravito-Heavisidian fields of dyons (gravito-dyons) was demonstrated by Rajput (1982, 1984; Rajput and Gunwant, 1989) and the relevant unified field equations were derived in a unique, consistent, and symmetrical way. A quaternion gauge theory of unified non-Abelian fields of dyons and gravito-dyons was constructed by Bisht *et al.* (1991*a*) and the corresponding quantum equations were derived.

Considering the resemblance between null tetrad components and 2×2 Pauli spin matrices, Bisht *et al.* (1991*b*) constructed the spinor equivalents of generalized fields of dyons and derived the relevant quantum equations of dyons by means of null tetrad notation. Extending the null tetrad formalism to the case of a non-Abelian Yang–Mills field, in the present paper we reformulate the generalized Yang–Mills potential and field strength of dyons in terms of null tetrad notation and derive the corresponding field equations in a unique and consistent way. Starting with a brief description of dyons and gravito-dyons, we combine the generalized fields of dyons and gravito-dyons and derive the corresponding field equations and equation of motion in a manifestly covariant way. The non-Abelian gauge theory of dyons and gravito-dyons is reformulated from two Yang–Mills gauge potentials associated with the electric and magnetic coupling parameters of dyons. The spinor equivalents of the generalized Yang–Mills potential and field strength antisymmetric tensor of dyons are constructed in terms of symmetric spinor components by means of null tetrad notations and spin coefficients. A null tetrad formulation of non-Abelian dyons is also constructed in flat space-time and corresponding Yang–Mills field equations are also derived. It is emphasized that two potential descriptions of dyons are compulsory in the Abelian case while in the non-Abelian case dyon solutions are obtained in terms of duality transformations between spinors, potentials, and fields. It is shown that dyon solutions exhibit both electric and magnetic charges.

2. GENERALIZED ELECTROMAGNETIC FIELDS OF DYONS

The generalized charge on dyons is defined as

$$q = e - ig \tag{2.1}$$

where e and g are, respectively, the electric and magnetic charge. Two four-potentials $\{A_\mu\} = \{Q^e, \mathbf{A}\}$ and $\{\mathbf{B}_\mu\} = \{Q^g, \mathbf{B}\}$ were introduced by Rajput *et al.* (1982, 1984, 1988) to avoid the use of arbitrary string variables, and are known as electric and magnetic four-potentials, respectively. These two potentials lead to the following covariant forms of the

Maxwell–Dirac equation (in terms of natural units $c = \hbar = 1$):

$$\begin{aligned} F_{\mu\nu,\nu} &= j_\mu \\ F^d_{\mu\nu,\nu} &= k_\mu \end{aligned} \qquad (\mu, \nu = 0, 1, 2, 3) \tag{2.2}$$

where $\{j_\mu\} = \{p^e, \mathbf{j}\}$ and $\{k_\mu\} = \{p^m, \mathbf{k}\}$ are, respectively, the electric and magnetic four-current source densities of dyons and

$$F_{\mu\nu} = E_{\mu\nu} - H^d_{\mu\nu} \tag{2.3a}$$

$$F^d_{\mu\nu} = H_{\mu\nu} + E^d_{\mu\nu} \tag{2.3b}$$

$$E_{\mu\nu} = A_{\mu\nu} - A_{\nu,\mu} \tag{2.3c}$$

$$H_{\mu\nu} = B_{\mu\nu} - B_{\nu,\mu} \tag{2.3d}$$

$$E^d_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}E^{\rho\sigma} \tag{2.3e}$$

$$H^d_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}H^{\rho\sigma} \tag{2.3f}$$

In equations (2.2) and (2.3) the symbol d denotes the dual part, a comma represents partial space-time derivative, $\epsilon_{\mu\nu\beta\sigma}$ is a 4-index Levi-Civita symbol, and μ, ν, β, and σ are space-time indices having values 0, 1, 2, 3. Assuming the generalized four-current and generalized four-potential of a dyon as

$$J_\mu = j_\mu - ik_\mu \tag{2.4}$$

$$V_\mu = A_\mu - iB_\mu \tag{2.5}$$

with the help of equation (2.2), we write the following generalized field equation of dyons:

$$G_{\mu\nu,\nu} = J_\mu \tag{2.6}$$

where

$$G_{\mu\nu} = F_{\mu\nu} - iF^d_{\mu\nu} \tag{2.7}$$

is called the generalized electromagnetic field tensor of dyons. Equations (2.2) and (2.6) are invariant under Lorentz transformations and duality transformations,

$$(F, F^d) = (F\cos\theta + F^d\sin\theta,\ -F\sin\theta + F^d\cos\theta) \tag{2.8a}$$

$$(A_\mu, B_\mu) = (A_\mu\cos\theta + B_\mu\sin\theta,\ -A_\mu\sin\theta + B_\mu\cos\theta) \tag{2.8b}$$

$$(j_\mu, k_\mu) = (j_\mu\cos\theta + k_\mu\sin\theta,\ -j_\mu\sin\theta + k_\mu\cos\theta) \tag{2.8c}$$

where we have used

$$\frac{g}{e} = \frac{B_\mu}{A_\mu} = \frac{k_\mu}{j_\mu} = -\tan\theta \tag{2.9}$$

so that the generalized charge of a dyon (2.1) may be expressed as

$$q = |q|e^{-i\theta} \tag{2.10}$$

In addition to dual symmetry, the covariant field equation (2.6) leads to the following conservation laws (Bisht *et al.*, 1991*b*; Rajput and Prakash Om, 1978):

(a) Invariance under a pure rotation in charge space or its combination with a transformation containing simultaneous space and time reflection (strong symmetry).

(b) A weak symmetry under charge reflection combined with space or time reflections (not both).

(c) A weak symmetry under *PT* (combined operation of parity and time reversal) and a strong symmetry under *CPT* (combined operation of charge conjugation, parity, and time reversal).

3. FIELDS ASSOCIATED WITH GRAVITO-DYONS

Postulating the existence of a dual mass associated with the gravimagnetic (Heavisidian) field playing the role of the monopole in the linear theory of gravitation and keeping in mind the difficulties faced by Dirac (1931) veto-like electromagnetism, Rajput *et al.* (1982, 1984; Bisht *et al.*, 1990; Rajput and Gunwant, 1989) describe a theory of particles carrying simultaneously gravitational and Heavisidian charges (i.e., gravito-dyons). The generalized charge of a gravito-dyon is defined as

$$q = m - ih(i = \sqrt{-1}) \tag{3.1}$$

where m and h are masses (charges) associated with gravitational and Heavisidian (g-magnetic) fields. Adopting the same method as in Section 2, we write the following tensorial form of the Maxwell–Dirac equation for gravito-dyons:

$$\begin{aligned} f_{\mu\nu,\nu} &= -j_\mu^{(\mathrm{G})} \\ f^d_{\mu\nu,\nu} &= -j_\mu^{(\mathrm{H})} \end{aligned} \tag{3.2}$$

where $\{J_\mu^{\mathrm{G}}\}$ and $\{J_\mu^{\mathrm{H}}\}$ are, respectively, gravitational and Heavisidian four-current source densities and

$$f_{\mu\nu} = L_{\mu\nu} - K^d_{\mu\nu} \tag{3.3a}$$

$$f^d_{\mu\nu} = K_{\mu\nu} + L^d_{\mu\nu} \tag{3.3b}$$

$$L_{\mu\nu} = a_{\mu,\nu} - a_{\nu,\mu} \tag{3.3c}$$

$$K_{\mu\nu} = b_{\mu,\nu} - b_{\nu,\mu} \tag{3.3d}$$

$$L^d_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}L^{\rho\sigma} \tag{3.3e}$$

$$K^d_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}K^{\rho\sigma} \tag{3.3f}$$

where $\{a_\mu\}$ and $\{b_\mu\}$ are, respectively, gravitational and Heavisidian (g-magnetic) four-potentials, and other symbols have their usual meaning as in Section 2. Like electromagnetism, the theory of gravito-dyons describes the existence of generalized charge, potential, current, and generalized field tensor as complex quantities with their real and imaginary parts as gravitational (g-electric) and Heavisidian (g-magnetic) constituents, i.e.,

$$v_\mu = a_\mu - ib_\mu \qquad \text{(generalized four-potential)} \tag{3.4}$$

$$s_\mu = j_\mu^{(\mathrm{G})} - ij_\mu^{(\mathrm{H})} \qquad \text{(generalized four-current)} \tag{3.5}$$

$$g_{\mu\nu} = f_{\mu\nu} - if_{\mu\nu}^d \qquad \text{(generalized field tensor)} \tag{3.6}$$

Equations (3.2) and (3.6) can be combined into the following covariant field (Maxwell–Dirac) equation for generalized gravito-Heavisidian fields of gravito-dyons:

$$g_{\mu\nu,\nu} = -s_\mu \tag{3.7}$$

Equation (3.7) is the generalized field equation (i.e., Maxwell–Dirac equation) in covariant form of gravito-dyons. This equation is invariant under Lorentz transformations, gauge transformations, and duality transformations and also follows the conservation laws, like the generalized electromagnetic fields of dyons given in Section 2. A gauge-covariant and rotationally symmetric theory of the angular momentum operator for gravito-dyons has already been described by Rajput (1982, 1984) in terms of structural symmetry between the generalized electromagnetic fields of dyons and the generalized gravito-Heavisidian fields of gravito-dyons. The duality transformation (2.8) also holds good for generalized fields of gravito-dyons in terms of structural symmetry. In other words, we interpret the results of the generalized theory of gravito-dyons as gravitational analogs of the generalized electromagnetic fields of dyons.

4. COMBINED GRAVITO-HEAVISIDIAN AND ELECTROMAGNETIC FIELDS OF DYONS

We are now in position to combine the generalized electromagnetic fields of dyons and the generalized gravito-Heavisidian fields of gravito-dyons by enlarging their Abelian gauge group structures. Let us start with the Lagrangian density of these two fields. The total Lagrangian density which couples the gravito-Heavisidian and electromagnetic fields of dyons may be written as

$$L = L_{\mathrm{EM}} + L_{\mathrm{GH}} \tag{4.1}$$

with

$$L_{\mathrm{EM}} = M - \tfrac{1}{8} G^{+}_{\mu\nu} G^{\mu\nu} + \tfrac{1}{2} V^{+}_{\mu} J^{\mu} + \mathrm{h.c.} \tag{4.2a}$$

and

$$L_{\mathrm{GH}} = M - \tfrac{1}{8} g^{+}_{\mu\nu} g^{\mu\nu} + \tfrac{1}{2} v^{+}_{\mu} s^{\mu} + \mathrm{h.c.} \tag{4.2b}$$

where h.c. denotes Hermitian conjugate, and the other terms have already been explained in Sections 2 and 3. The effective mass M of dyons and gravito-dyons is defined as

$$M_{\mathrm{eff}} = m - (\alpha - 1)/2h \tag{4.3}$$

where $\alpha = +1$ for generalized electromagnetic fields and $\alpha = -1$ for generalized gravito-Heavisidian fields of dyons. The Lagrangian density given by (4.1) yields the field equations (2.6) for generalized electromagnetic fields and equation (3.7) for generalized gravito-Heavisidian fields of dyons. The Lagrangian density (4.1) also yields the following forms of the Lorentz force equation of motion respectively for generalized electromagnetic and generalized gravito-Heavisidian fields:

$$M\ddot{x}_{\mu} = \mathrm{Re}(q^{*} G_{\mu\nu} U^{\nu}) \quad \text{(generalized electromagnetic fields)} \tag{4.4}$$

and

$$M\ddot{x}_{\mu} = \mathrm{Re}(Q^{*} g_{\mu\nu} u^{\nu}) \quad \text{(generalized gravito-Heavisidian fields)} \tag{4.5}$$

where $\{u^{\nu}\}$ is four-velocity of particles, $\ddot{x}_{\mu}$ is the four-acceleration, and Re denotes the real part. We can write equations (4.1) and (4.2) as

$$L = -M - \tfrac{1}{8}(g^{+}_{\mu\nu} g^{\mu\nu} + G^{+}_{\mu\nu} G^{\mu\nu}) + \tfrac{1}{2}(v^{+}_{\mu} s^{\mu} + V^{+}_{\mu} J^{\mu}) + \mathrm{h.c.} \tag{4.6}$$

and the combined field equations and Lorentz forces as

$$\partial^{\nu}(G_{\mu\nu} + g_{\mu\nu}) = (J_{\mu} - s_{\mu}) \tag{4.7}$$

$$M\ddot{x}_{\mu} = \mathrm{Re}(Q^{*} g_{\mu\nu} u^{\nu} + q^{*} G_{\mu\nu} U^{\nu}) \tag{4.8}$$

In deriving the equations of motion (4.8) we have taken into account the standard relation between four-velocity, four-current, and the respective charges of the generalized fields. To put the gravito-Heavisidian and electromagnetic fields of the dyons in a compact form in terms of single charge and potential, etc., let us define the combined generalized charge of these two dyons as

$$Z = q - jQ \tag{4.9}$$

where $j = \sqrt{-1}$ is an imaginary quantity other than $i = \sqrt{-1}$ given in Sections 2 and 3, q and Q are, respectively, given by equations (2.1) and

(3.1), and $i \cdot j$ follow the relations

$$ij = -ji = k \quad \text{(say)}$$

and

$$\begin{aligned} i(ij) &= (ii)j = -j \\ (ij)j &= i(jj) = -i \\ ik &= -ki = -j \\ kj &= -jk = i \\ i^2 &= j^2 = k^2 = -1 \end{aligned} \tag{4.10}$$

As such, equation (4.9) is a quaternion with basis elements $(1, i, j, k)$ and the combined electromagnetic and gravito-Heavisidian fields are quaternion-valued quantities,

$$\begin{aligned} Z &= (e - ig) - j(m - ih) \\ &= e - ig - jm - kh \end{aligned} \tag{4.11}$$

Similarly, we write the potential, field tensor, and current source densities of the combination of generalized fields of dyons and gravito-dyons as

$$C_\mu = A_\mu - iB_\mu - ja_\mu - kb_\mu \tag{4.12}$$

$$W_{\mu\nu} = F_{\mu\nu} - iF^d_{\mu\nu} - jf_{\mu\nu} - kf^d_{\mu\nu} \tag{4.13}$$

$$j_\mu = j_\mu - ij_\mu - jj^{(G)}_\mu - kj^{(H)}_\mu \tag{4.14}$$

The quaternion conjugate charge of combined electromagnetic and gravito-Heavisidian fields of dyons is defined as

$$\bar{Z} = e + ig + jm + kh \tag{4.15}$$

The Lagrangian density may be written as follows in a compact, simpler, and consistent form:

$$L = -M - \tfrac{1}{8} W_{\mu\nu} G_{\rho\sigma} \eta^{\mu\nu} \eta^{\rho\sigma} + \tfrac{1}{2} C_\mu j_\nu \eta^{\mu\nu} + \text{q.e.} \tag{4.16}$$

where q.e. denotes quaternion conjugation values and $\eta^{\mu\nu}$ is the flat-space time metric with signature 2. The combined gauge-variant momentum then may be written as

$$\begin{aligned} P_\mu &\to p_\mu - \tfrac{1}{2}\bar{Z} c_\mu + \text{q.e.} \\ &= p_\mu - eA_\mu - gB_\mu - ma_\mu - hb_\mu \end{aligned} \tag{4.17}$$

which leads to the equation of motion (4.8) into the following compact and simpler form:

$$\begin{aligned} M\ddot{x}_\mu &= (\bar{Z}W_{\mu\nu} + \text{q.e.})u^\nu \\ &= \tfrac{1}{2}\bar{Z}W_{\mu\nu}u^\nu + \text{q.e.} \end{aligned} \tag{4.18}$$

The Lagrangian density (4.16) also yields equation (4.18) and the following convenient simple and compact form of the field equation (3.14):

$$W_{\mu\nu,\nu} = \mathscr{F}_\mu \tag{4.19}$$

Here we have combined the generalized electromagnetic and generalized gravito-Heavisidian fields into simple and compact quaternionic forms.

5. NON-ABELIAN GAUGE THEORY OF DYONS

Now we extend our theory of dyons given in Section 2 for the Abelian case to non-Abelian Yang–Mills gauge theory. The invariance of the generalized field equations of dyons in non-Abelian gauge theory is obtained under local gauge transformations:

$$\psi \to \psi' = S^{-1}\psi \tag{5.1}$$

where S is the local gauge group element of a 2×2 unitary unimodular $SU(2)$ isospin rotation group and describes the isospin doublet. Identifying the generalized potential V_μ of dyons as the Yang–Mills potential in terms of 2×2 Hermitian matrices in internal gauge space, all the derivatives of ψ are defined as covariant derivatives,

$$\nabla_\mu = \partial_\mu - q^* V_\mu(\times) \tag{5.2}$$

where the vectors and cross product ($\times$) are defined in internal space, and q is a coupling constant. The non-Abelian gauge field strength is therefore defined as

$$\begin{aligned} \mathscr{G}_{\mu\nu} &= \mathbf{V}_{\mu\nu} - \mathbf{V}_{\nu\mu} + iq^*[\mathbf{V}_\mu, \mathbf{V}_\nu] \\ &= \mathbf{G}_{\mu\nu} + iq^*[\mathbf{V}_\mu, \mathbf{V}_\nu] \end{aligned} \tag{5.3}$$

where

$$\mathbf{G}_{\mu\nu} = \mathbf{V}_{\mu,\nu} - \mathbf{V}_{\nu,\mu} \tag{5.4}$$

For gravito-dyons the coupling constant q plays the role of generalized mass, while in general it is identified as the generalized charge or isospin charge in the Yang–Mills gauge space.

In the internal two-dimensional complex space introduced at each point of Minkowski space-time, the charged field described by ψ in $SU(2)$

is replaced by $\exp\{i\Lambda^0(x)\}$ is $SU(2) \times U(1)$, where $\Lambda^0(x)$ is a phase factor for $U(1)$. Then the basis spinors of this internal space are acted upon by the following elements of $SU(3)$:

$$\tilde{S} = \tilde{S}(x) \exp[-i\Lambda^0(x)] \tag{5.5}$$

under this gauge transformation, the 2×2 matrix potential $\mathbf{V}_\mu$ and the matrix field tensor $\mathbf{G}_{\mu\nu}$ transform as

$$\mathbf{V}'_\mu = \tilde{S}^{-1}\mathbf{V}_\mu \tilde{S} - \tilde{S}^{-1}\,\partial_\mu \tilde{S} \tag{5.6}$$

and

$$\mathbf{G}'_{\mu\nu} = \tilde{S}^{-1}\mathbf{G}_{\mu\nu}\tilde{S} \tag{5.7}$$

Instead of matrices V_μ and $G_{\mu\nu}$, we may define the gauge potential $V_{a\mu}$ and the gauge field strength $G_{a\mu\nu}$ as

$$V_\mu = V_{a\mu}\Gamma^a \tag{5.8}$$

$$G_{\mu\nu} = G_{a\mu\nu}\Gamma^a \tag{5.9}$$

where repeated indices are summed over 1, 2, 3. The matrices Γ_a describe the infinitesimal generators of the group $SU(2)$ and satisfy the commutation relation

$$[\Gamma_a, \Gamma_b] = i\epsilon_{abc}\Gamma_c \tag{5.10}$$

Let us write equation (5.3) as

$$\mathcal{G}_{a\mu\nu} = \partial_\nu V_{a\mu} - \partial_\mu V_{a\nu} + q^*\epsilon_{abc}V_{b\nu}V_{c\mu} \tag{5.11}$$

$$\mathcal{G}_{a\mu\nu} = G_{a\mu\nu} + q^*\epsilon_{abc}V_{b\nu}V_{c\mu} \tag{5.12}$$

The covariant derivative of the field tensor $G_{\mu\nu}$ is given as

$$\nabla^\nu \mathcal{G}_{\mu\nu} = \partial^\nu \mathbf{G}_{\mu\nu} + iq^*\mathbf{V}^\nu \times \mathbf{G}_{\mu\nu} = \mathbf{J}_\mu \tag{5.13}$$

or

$$\nabla^\nu \mathcal{G}_{a\mu\nu} = \partial^\nu G_{a\mu\nu} + q^*\epsilon_{abc}V^{b\nu}G_{c\mu\nu} = J_{a\mu} \tag{5.14}$$

where

$$\mathbf{J}_\mu = \mathbf{j}_\mu + iq^*\mathbf{V}^\nu \times \mathbf{G}_{\mu\nu} \tag{5.15}$$

and

$$J_{a\mu} = j_{a\mu} + q^*\epsilon_{abc}V^{b\nu}G_{c\mu\nu} \tag{5.16}$$

are the generalized four-current with field associated with dyons. From equations (5.15)–(5.16) we get the conserved Noetherian current as

$$\mathbf{j}_\mu = \mathbf{J}_\mu - iq^*\mathbf{V}^\nu \times \mathbf{G}_{\mu\nu} \tag{5.17}$$

which gives

$$\partial^{\mu}\mathbf{j}_{\mu}=0 \tag{5.18}$$

while

$$\nabla^{\mu}\mathbf{J}_{\mu}=0. \tag{5.19}$$

These equations show that Noetherian current j_{μ} is conserved while the generalized non-Abelian current J_{μ} is not, but satisfies a generalized conservation law on replacing the ordinary derivative by the covariant derivative.

6. NONABELIAN GAUGE THEORY OF DYONS

In order to extend the theory of generalized gravito-Heavisidian fields of gravito-dyons to the non-Abelian case and to apply the Yang–Mills method to the gravitational field, the use of spinors is usually considered. Though our theory of gravito-dyons resembles the generalized electromagnetic fields of dyons, still it is hard to describe the isospinors with this theory, as we do not have the known internal quantum number in the linear theory of gravitation. Therefore we apply the method of spinors to extend the Yang–Mills method to generalized gravito-Heavisidian fields of gravito-dyons. For this case we consider the non-Riemannian space-time of Einstein's nonsymmetric theory associated with n-dimensional space. In this case we identify the symmetric part as pure gravitation, and the antisymmetric part subjected to an electromagnetic field as gravi-electromagnetic [i.e., field coupled by gravito-Heavisidian charge (mass)]. Analogous to the theory of dyons, the generalized four-potential of gravito-dyons is defined as

$$\tilde{V}_{\mu}=\tilde{A}_{\mu}-i\tilde{B}_{\mu}=\tilde{b}_{\mu}-iQ^{*}\tilde{a}_{\mu} \tag{6.1}$$

with

$$\tilde{A}_{\mu}=\tilde{b}_{\mu}+h\tilde{I}\tilde{a}_{\mu} \tag{6.2a}$$

and

$$\tilde{B}_{\mu}=-m\tilde{I}\tilde{a}_{\mu} \tag{6.2b}$$

where $\tilde{b}_{\mu}$ is the usual space-time gravitational four-vector potential and $\tilde{a}_{\mu}$ is the gravito-electromagnetic potential subjected to the antisymmetric part of the nonsymmetric metric. q, m, and h are described in Section 3 as generalized charge, gravitational, and Heavisidian charges. Four-potentials $\tilde{A}_{\mu}$ and $\tilde{B}_{\mu}$ are visualized as gravitational and Heavisidian potentials. These two four-potentials may be interpreted as potentials describing the coupling of the Heavisidian monopole with the gravitational field and the coupling of gravitational mass with the gravito-electromagnetic potential; the generalized potential V_μ is described here in terms of 2×2 spin matrices. Adopting the same procedure of generalized electromagnetic fields of dyons, we describe the invariance under the transformation matrix

$$\tilde{S}(x) = \tilde{S}(x) \exp\{-i\Lambda^0(x)\} \tag{6.3}$$

where $\tilde{S}(x)$ is the local element of group $SL(2, c)$ and requires the following transformation of generalized potential matrix V_μ:

$$V'_\mu = \tilde{S}^{-1}(x) V_\mu \tilde{S}(x) - \tilde{S}^{-1}\partial_\mu \tilde{S} \tag{6.4}$$

which yields

$$b'_\mu = \tilde{S}^{-1}(x) b_\mu \tilde{S}(x) - \tilde{S}^{-1}\partial_\mu \tilde{S} \tag{6.4a}$$

$$a'_\mu = a_\mu + |Q|^{-1}\partial_\mu \Lambda^{0'} \tag{6.4b}$$

where $Q = (m^2 + h^2)^{1/2}$. The transformations of potentials $A_\mu(x)$ and $B_\mu(x)$ may then be written as

$$\begin{aligned} A'_\mu(x) &= \tilde{S}^{-1} A_\mu(x) \tilde{S} - \tilde{S}^{-1}\, \partial_\mu \tilde{S} - \tan\theta\, \tilde{S}^{-1} B_\mu(x) \tilde{S} - I \sin\theta\, \partial_\mu \Lambda^{0'}(x) \\ B'_\mu(x) &= B_\mu(x) - I \cos\theta\, \partial_\mu \Lambda^{0'}(x) \end{aligned} \tag{6.5}$$

where $\theta = \tan^{-1} h/m$.

The generalized field tensor for gravito-dyons is then defined as

$$G_{\mu\nu} = \partial_\nu V_\mu - \partial_\mu V_\nu + iQ^*(V_\mu, V_\nu) \tag{6.6}$$

which is a 2×2 complex traceless matrix. Here Q^* is the complex conjugate of the generalized mass Q of gravito-dyons. This field strength may also be written as

$$G_{\mu\nu} = G_{\mu\nu} + iQ^* I f_{\mu\nu} \tag{6.7}$$

where

$$G_{\mu\nu} = \partial_\nu b_\mu - \partial_\mu b_\nu + iq^*(b_\mu, b_\nu) \tag{6.8}$$

and

$$f_{\mu\nu} = \partial_\nu a_\mu - \partial_\mu a_\nu \tag{6.9}$$

On taking the covariant derivative of equation (6.7), one can write the following expression for field equations of gravito-dyons:

$$\nabla^\nu G_{\mu\nu} = {}^\nu G_{\mu\nu} + iq^* V^\nu \times G_{\mu\nu} = J_\mu \tag{6.10}$$

where for brevity we have taken the Einstein gravitational constant ($k = 1$) and J_μ is the generalized current representing the source of the field of gravito-dyons, defined as

$$J_\mu = j_\mu + iq^* V^\nu \times G_{\mu\nu} = J_\mu \tag{6.11}$$

where

$$j_\mu = j_\mu^{(G)} - ij_\mu^{(H)} \tag{6.12}$$

and yields the following conservation law:

$$\nabla^\mu J_\mu = 0 \tag{6.13}$$

In equation (6.12) j_μ, the generalized current, is the combination of the gravitational and Heavisidian currents obtained from the corresponding potentials.

7. NULL TETRAD FORMULATION OF NON-ABELIAN DYONS AND GRAVITO-DYONS

Now we reformulate the non-Abelian gauge theory of dyons (given in Section 5) by means of the null tetrad notation of Newmann and Penrose (1962). Recently we have reformulated (Rajput *et al.*, 1991*b*) the generalized fields of dyons (given in Section 2) in terms of null tetrad notation by writing spinor-equivalent forms of corresponding fields, potentials, currents, equations of motion, and gauge-invariant-cum rotationally symmetric angular momentum operators with the extension of the Abelian gauge group $U(1)$ to the $SL(2, c)$ gauge group. Here also we follow the approach and notations of Carmeli (1977*a*, 1982), who studied the $SL(2, c)$ gauge theory of gravitation by enlarging the gauge group $SL(2, c) \times U(1) \times V'(1)$ to accommodate gravitation, electromagnetism, and monopoles. The difference between Carmeli and Huleihil's (1984) theory for applying the Newmann–Penrose method to the Yang–Mills field and the theory we are going to present here is that the former deals with the study of Yang–Mills gauge theory directly modeled from electromagnetism, while the latter describes the two-potential theory of dyons. In other words, our theory of Yang–Mills fields (given in Section 5) is modeled from the generalized electromagnetic theory (Section 2) of dyons instead of usual electromagnetism. The generalized fields, potentials, currents, and other quantum equations for dyons presented here are self-dual.

Following Newmann and Penrose (1962) and Carmeli (1977*a*,*b*, 1982; Carmeli and Huleihil, 1984; Carmeli *et al.*, 1989), we define a null tetrad of basis vectors l_μ, n_μ, m_μ, and $\bar{m}_\mu$ introduced at each point of the four-dimensional Riemannian manifold with signature 2. The tetrad is made from two real null vectors, l_μ and n_μ, and a pair of complex null vectors, m_μ and $\bar{m}_\mu$. The null tetrad basis vectors l_μ, n_μ, m_μ, and $\bar{m}_\mu$ are defined in terms of the orthonormal basis $(U_\mu, V_\mu, W_\mu, Z_\mu)$, where V_μ is timelike while the other three are spacelike vectors, i.e.,

$$l_\mu = 1/\sqrt{2}(U_\mu + V_\mu) \tag{7.1}$$

$$n_\mu = 1/\sqrt{2}(U_\mu - V_\mu) \tag{7.2}$$

$$m_\mu = 1/\sqrt{2}(W_\mu + iZ_\mu) \tag{7.3}$$

$$\bar{m}_\mu = 1/\sqrt{2}(W_\mu - iZ_\mu) \tag{7.4}$$

These null tetrads satisfy the following pseudo-orthogonality relations:

$$\begin{aligned} l^\mu n_\mu &= -m^\mu \bar{m}_\mu = 1 \\ l^\mu l_\mu &= m^\mu m_\mu = n^\mu n_\mu = \bar{m}^\mu \bar{m}_\mu = 0 \end{aligned} \tag{7.5}$$

The generic symbol $Z^\mu p$ for the null tetrads $(l_\mu, m_\mu, n_\mu, \bar{m}_\mu)$ with $p = 0, 1, 2, 3$ yields the following expression for the contravariant components of the metric tensor:

$$\begin{aligned} g_{\mu\nu} &= Z^\mu_p Z^\nu_q \eta^{pq} \\ &= 2[l^{\langle\mu} n^{\nu\rangle} - m^{\langle\mu} \bar{m}^{\nu\rangle}] \\ &= l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu \end{aligned} \tag{7.6}$$

where η^{pq} is the flat space-time metric,

$$\eta^{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \eta_{pq} \tag{7.7}$$

and is used to raise and lower the tetrad indices, i.e.,

$$l_\mu = g_{\mu\nu} l^\nu; \qquad n_\mu = g_{\mu\nu} n^\nu; \qquad m_\mu = g_{\mu\nu} m^\nu; \qquad \bar{m}_\mu = g_{\mu\nu} \bar{m}^\nu \tag{7.8}$$

The internal space has been chosen as a two-dimensional complex space spanned by a pair of spinors and the gauge field equations are the Newmann–Penrose (1962) equations. One of the advantages of the $SL(2, c)$ approach of Carmeli (1977*a*, 1982) is the possibility of deriving equations for Newmann–Penrose spin coefficients in terms of null tetrad basis vectors and their intrinsic covariant derivatives. The $SL(2, c)$ matrix G, whose elements are written in the following manner, is made up from the two basis spinors G^A_a $(a, A = 0, 1)$:

$$G = \|G^A_0\| = \begin{pmatrix} l^0 & l^1 \\ n^0 & n^1 \end{pmatrix} \tag{7.9}$$

The variable $\sigma_{\mu ab'}$, where σ_j are 2×2 Pauli spin matrices for $j = 1, 2, 3$, and σ_0 is a unit matrix, are defined in terms of null tetrad vectors as

$$\sigma^\mu = \|\sigma^\mu_{ab'}\| = \begin{pmatrix} l^\mu & m^\mu \\ \bar{m}^\mu & n^\mu \end{pmatrix} \qquad (a, b' = 0, 1) \tag{7.10}$$

The intransic derivatives (i.e., the directional covariant derivatives) are defined as elements of the matrix,

$$\begin{aligned} \sigma^\mu \nabla_\mu &= \|\nabla_{ab'}\| = \|\sigma^\mu_{ab'} \nabla_\mu\| \\ &= \begin{pmatrix} D & \delta \\ \bar{\delta} & \nabla \end{pmatrix} = \begin{pmatrix} \nabla_{00'} & \nabla_{01'} \\ \nabla_{10'} & \nabla_{11'} \end{pmatrix} \end{aligned} \tag{7.11}$$

where

$$D = \nabla_\mu l^\mu = l^\mu \partial_\mu \tag{7.12a}$$

$$\nabla = \nabla_\mu n^\mu = n^\mu \partial_\mu \tag{7.12b}$$

$$\delta = \nabla_\mu m^\mu = m^\mu \partial_\mu \tag{7.12c}$$

$$\bar{\delta} = \nabla_\mu \bar{m}^\mu = \bar{m}^\mu \partial_\mu \tag{7.12d}$$

On the other hand, the ordinary derivative ∂_μ is defined as

$$\partial_\mu = \sigma^{ab'}_\mu \partial_{ab'} \tag{7.13}$$

The dyad coefficients of the matrices are given by

$$C_{ab'} = \sigma^\mu_{ab'} C_\mu \tag{7.14}$$

where the four-matrices $C_{ab'}$ have the form

$$\begin{aligned} C_{00'} &= \begin{pmatrix} \epsilon & k \\ \pi & -\epsilon \end{pmatrix}; & C_{01'} &= \begin{pmatrix} \beta & -\sigma \\ \mu & -\beta \end{pmatrix} \\ C_{10'} &= \begin{pmatrix} \sigma & -p \\ \lambda & -\alpha \end{pmatrix}; & C_{11'} &= \begin{pmatrix} \tau & -T \\ v & -T \end{pmatrix} \end{aligned} \tag{7.15}$$

and $C^+_{ab'}$ is the Hermitian conjugate of $C_{ab'}$. The 12 elements ϵ, k, π, etc., of the matrices $C_{ab'}$ are functions that were first introduced by Newmann and Penrose and are called the spin coefficients.

Before applying the null tetrad method to Yang–Mills generalized potentials and fields of dyons, respectively, given by equations (5.8) and (5.9), let us decompose them into real and imaginary parts,

$$\text{Re } V_\mu = A_\mu = A_{\mu a} \Gamma^a = p_{\mu a} \Gamma^a \tag{7.16a}$$

$$\text{Im } V_\mu = B_\mu = B_{\mu a} \Gamma^a = q_{\mu a} \Gamma^a \tag{7.16b}$$

$$\text{Re } G_{\mu\nu} = E_{\mu\nu} = F_{\mu\nu a} \Gamma^a = f_{\mu\nu a} \Gamma^a \tag{7.16c}$$

$$\text{Im } G_{\mu\nu} = H^d_{\mu\nu} = F^d_{\mu\nu a} \Gamma^a = h_{\mu\nu a} \Gamma^a \tag{7.16d}$$

where

$$f_{\mu\nu a} = p_{\mu a,\nu} - p_{\nu a,\mu} + q_1 \epsilon_{abc} p_{\nu b} p_{\mu c} \tag{7.17a}$$

and

$$h_{\mu\nu a} = q_{\mu a,\nu} - q_{\nu a,\mu} + q_2 \epsilon_{abc} q_{\nu b} q_{\mu c} \tag{7.17b}$$

where μ, ν are space-time indices and a, b, c are Yang–Mills indices for internal symmetry; $G_{\mu\nu}$ is the Yang–Mills gauge field strength modeled from the dynamics of electric charge and obtained from its gauge potential A_μ, and $F^d_{\mu\nu a}$ is also the Yang–Mills field strength obtained from another gauge potential B_μ responsible for the dynamics of magnetic charge. These two Yang–Mills gauge field strengths and the corresponding gauge potentials are coupled to the complex gauge field and gauge potential of dyons by equations (5.8) and (5.9). Let us define the following spinor-equivalent (i.e., tetrad component) forms of Yang–Mills potentials and fields as:

$$p_{ac'k} = \sigma^{\mu}_{ac'} p_{\mu k} \tag{7.18a}$$

$$f_{ab'cd'k} = \sigma^{\mu}_{ab'} \sigma^{\nu}_{cd'} f_{\mu\nu k} \tag{7.18b}$$

$$q_{ac'k} = \sigma^{\mu}_{ac'} q_{\mu k} \tag{7.18c}$$

$$h_{ab'cd'k} = \sigma^{\mu}_{ab'} \sigma^{\nu}_{cd'} f^{d}_{\mu\nu k} \tag{7.18d}$$

where a, b', c, d' are spinor indices taking the values 0, 1 and 0′, 1′; k is the Yang–Mills index taking values 1, 2, 3 for the $SU(2)$ group denoting the vector components in internal space; and μ, ν are space-time indices taking values 0, 1, 2, 3 in Minkowski space-time.

Equations (7.18a) and (7.18c) can be combined into the complex form of the generalized potential of dyons,

$$V_{ac'k} = \sigma^{\mu}_{ac'} V_{\mu k} \tag{7.19a}$$

$$V_{\mu k} = P_{\mu k} - i q_{\mu k} \tag{7.19b}$$

Similarly, (7.18b) and (7.18d) couple to the following spinor-equivalent form of the generalized field strength of dyons:

$$G_{ab'cd'k} = \sigma^{\mu}_{ab'} \sigma^{\nu}_{cd'} G_{\mu\nu k} \tag{7.20}$$

$$G_{\mu\nu k} = f_{\mu\nu k} - i h_{\mu\nu k} \tag{7.20b}$$

We are now able to decompose real Maxwell antisymmetric tensors with tetrad components in the following forms:

$$f_{ab'cd'} = \epsilon_{ac} \bar{\phi}_{b'd'} + \phi_{ac} \epsilon_{b'd'} \tag{7.21}$$

$$h_{ab'cd'} = \epsilon_{ac} \bar{\psi}_{b'd'} + \psi_{ab} \epsilon_{b'd'} \tag{7.22}$$

where

$$\phi_{ac} = \phi_{ca} = \tfrac{1}{2} f^{b'}_{ab'c} \tag{7.23}$$

$$\psi_{ac} = \psi_{ca} = \tfrac{1}{2} h^{b'}_{ab'c} \tag{7.24}$$

and $\phi_{b'd'}$ and $\psi_{b'd'}$ are complex conjugates of ϕ_{bd} and ψ_{bd} which may be called the electromagnetic field spinors associated with electric and magnetic charges. In equations (7.21) and (7.22) ϵ_{ac} and $\epsilon_{b'd'}$ (and ϵ^{ac}, $\epsilon^{b'd'}$) are the Levi-Civita antisymmetric spinors given by

$$\epsilon_{ac} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{ac} = i\sigma^2 \tag{7.25}$$

The spinor indices can be raised or lowered with the help of these spinors as

$$\xi^a = \epsilon^{ac} \xi_c \qquad (\xi^a \text{ is a 2-component spinor}) \tag{7.26a}$$

$$\xi_c = \xi^a \epsilon_{ac} \tag{7.26b}$$

which immediately shows

$$\xi^a_a = -\xi^a_a \tag{7.27}$$

For every pair of indices we may define a symmetric and antisymmetric part:

$$\xi_{ab} = \tfrac{1}{2} \xi_{ab} \xi^c_c + \tfrac{1}{2} \xi_{\tilde{a}\tilde{b}} \tag{7.28}$$

The tilde over the indices a and b indicates that a symmetric transformation is to be performed as $\xi_{\tilde{a}\tilde{b}} = \xi_{ba} + \xi_{ab}$.

Equations (7.18b) and (7.18d) may directly be expanded in the following forms in terms of their null tetrad components, on substitution of equations (7.11)–(7.15):

$$\begin{aligned} f_{ab'cd'k} = {} & \partial_{cd'} p_{ab'k} - \partial_{ab'} p_{cd'k} - |A_{cd'}|^r_a p_{rb'k} \\ & - p_{ar'k} |A^+_{d'c}|^{r'}_b + (A_{ab'})^r_c p_{rd'k} \\ & + p_{cr'k} (A^+_{b'a})^{r'}_{d} + g_1 \epsilon_{klm} p_{ab'm} p_{cd'l} \end{aligned} \tag{7.29}$$

and

$$\begin{aligned} h_{ab'cd'k} = {} & \partial_{cd'} q_{ab'k} - \partial_{ab'} q_{cd'k} - |B_{cd'}|^r_a q_{rb'k} \\ & - q_{ar'k} |B^+_{d'c}|^{r'}_b + (B_{ab'})^r_c q_{rd'k} \\ & + q_{cr'k} (B^+_{b'a})^{r'}_{d} + g_2 \epsilon_{klm} q_{ab'm} q_{cd'l} \end{aligned} \tag{7.30}$$

In analogy with the decomposition of Maxwell's field tensors given by equations (7.21) and (7.22), we may decompose the Yang–Mills field strengths $f_{ab'cd'k}$ and $h_{ab'cd'k}$ in terms of dyad components χ_{abk} and ψ_{abk} of the symmetric spinors χ_{ABK} and ψ_{ABK} as

$$f_{ac'bd'k} = \epsilon_{ab}\bar{\chi}_{c'd'k} + \epsilon_{c'd'}\chi_{abk} \tag{7.31a}$$

$$h_{ac'bd'k} = \epsilon_{ab}\psi_{c'd'k} + \epsilon_{c'd'}\psi_{abk} \tag{7.31b}$$

Generalized Yang–Mills fields and potential associated with dyons can then be written as follows in terms of complex quantities:

$$q_{ab'cd'k} = f_{ab'cd'k} - ih_{ab'cd'k} = \sigma^{\mu}_{ab'}\sigma^{\nu}_{cd'}q_{\mu\nu k} \tag{7.32a}$$

$$V_{ac'k} = p_{ac'k} - iq_{ac'k} = \sigma^{\mu}_{ac'}V_{\mu k} \tag{7.32b}$$

Equation (7.32a) is then written in terms of the following null tetrad notation:

$$\begin{aligned} q_{ab'cd'k} = {} & \partial_{cd'}q_{ab'k} - \partial_{ab'}q_{cd'k} - \left|V_{cd'}\right|^{r}_{a}q_{rb'k} \\ & - q_{ar'k}\left|V^{+}_{d'c}\right|^{r'}_{b} + (V_{ab'})^{r}_{c}q_{rd'k} \\ & + q_{cr'k}(V^{+}_{b'a})^{r'}_{d} + q^{*}\epsilon_{klm}q_{ab'm}q_{cd'l} \end{aligned} \tag{7.33}$$

where $V = A - iB$,

$$q = q_1 - iq_2 \tag{7.34}$$

and

$$q_{ac'bd'k} = \epsilon_{ab}\phi_{c'd'k} + \epsilon_{cd'}\phi_{abk} \tag{7.35}$$

$$\phi = \chi - i\psi \tag{7.36}$$

The spinor ϕ_{abk} is symmetric in its spinor indices a and b and consists of 3×3 complex components $\phi_{00k'}$, $\phi_{01k} = \phi_{10k}$, and ϕ_{11k} with $k = 1, 2, 3$. These nine components are again complex for the generalized theory of dyons and their real and imaginary constituents are equivalent to the real components of electric field strength $f_{\mu\nu k}$ and magnetic field strength $h_{\mu\nu k}$. Thus the components $\chi_{00k'}$, $\chi_{01k} = \chi_{10k}$, χ_{11k}, ψ_{00k}, $\psi_{01k} = \psi_{10k}$, and ψ_{11k} are real-valued spinor components of the generalized spinor ϕ associated with generalized electromagnetic fields of dyons. In the light of equations (7.21), (7.30), and (7.31) we can write the components χ, ψ, and ϕ as

$$\begin{aligned} \chi_{0k} &= \chi_{00k} \\ &= (\delta - \beta + \bar{\pi} - \bar{\alpha})p_{00'k} - (D + \bar{\epsilon} - \epsilon - \bar{p})p_{01'k} + p_{10'k} - kp_{11'k} \\ &\quad + g_1\epsilon_{jlm}p_{00'm}p_{01'l} \end{aligned} \tag{7.37}$$

$$
\begin{aligned}
\chi_{1k} &= \chi_{01k} \\
&= \chi_{10k} \\
&= \tfrac{1}{2}(\nabla - \bar{\mu} + \mu - \tau - \bar{\tau})p_{00'k} - (\bar{\delta} + \alpha - \pi - \bar{T} - \bar{\beta})p_{01'k} \\
&\quad + (\delta + T + \beta + \pi - \bar{\alpha})p_{10'k} - (D + p - \bar{p} + \epsilon + \bar{\epsilon})p_{11'k} \\
&\quad + g_1\epsilon_{klm}(p_{00'm}p_{11'l} - p_{01'm}p_{10'l})
\end{aligned}
\tag{7.38}
$$

$$
\begin{aligned}
\chi_{2k} &= \chi_{11k} \\
&= -\nu p_{00'k} + \lambda p_{01'k} + (\nabla + \nu - \bar{\nu} + \bar{\mu})p_{10'k} \\
&\quad - (\bar{\delta} + \alpha + \bar{\beta} - \bar{T})p_{11'k} + g_1\epsilon_{klm}p_{10'm}p_{11'l}
\end{aligned}
\tag{7.39}
$$

Similarly, the expressions for ψ_{0k}, ψ_{1k}, and ψ_{2k} can be written explicitly; the Yang–Mills field equation for the generalized gauge theory of dyons is given by equation (5.13). This equation is self-dual in terms of duality transformations given in Section 2. The spinor-equivalent form of the Yang–Mills field equation (5.13) is written as

$$
\nabla_{AB'}G_k^{CD'AB'} = -g\epsilon_{klm}V^{AB'l}G^{CD'AB'm} + J_k^{CD'} \tag{7.40}
$$

Thus we have

$$
G_k^{CD'AB'} = 2\epsilon^{D'B'}\chi_k^{CA} \tag{7.41}
$$

on using the self-duality of spinors, i.e.,

$$
\chi \to \chi; \qquad \psi \to -i\psi \qquad \text{so that} \qquad \phi \to \phi = 2^{\psi} \tag{7.42}
$$

while in general χ and ψ follow the duality transformation equation (2.8). Thus equation (7.40) reduces to the following expression for the generalized Yang–Mills field equation of dyons:

$$
\epsilon^{D'B'}\nabla_{AB'}/^{CA} = -g\epsilon_{klm}V^{AB'l}G^{CD'AB'm} + J_k^{CD'} \tag{7.43}
$$

This equation resembles the Yang–Mills field equation of Carmeli (1977*a*, 1982) on imposition of the following self-duality conditions for dyonic fields:

$$
\begin{aligned}
&A_\mu \to A_\mu; \qquad B_\mu = -iA_\mu \;\Rightarrow\; V_\mu = A_\mu \\
&F_{\mu\nu} = F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}; \qquad F^d_{\mu\nu} = F^d_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \\
&G_{\mu\nu} = F_{\mu\nu} - iF^d_{\mu\nu}
\end{aligned}
\tag{7.44}
$$

so that

$$f_k^{(+)CD'AB'} = G_k^{CD'AB'} = 2\epsilon^{D'B'}\chi_k^{CA}$$

From the foregoing analysis Carmeli's (1977*a*, 1982) equations can be written by letting the magnetic potential vanish. Unfortunately, one-potential theory is not enough to describe particles carrying the simultaneous existence of electric and magnetic charges, free from Dirac (1931) string singularities in Abelian gauge theory. To remove arbitrary string variables, a two-potential description of dyons is essential. On the other hand, in self-dual Yang–Mills non-Abelian gauge theory, where monopoles appear as classical solutions of gauge field equations, one is free to choose a one-potential description of dyons. But if one defines Yang–Mills theory as the direct modeling of electromagnetism, two-potential description of dyons in Yang–Mills gauge theory not only solves the singularity problem of Abelian gauge theory, but also leads to dual dynamics between electric and magnetic charges in internal space where they appear as the classical solutions of Yang–Mills field equations. Second, the dyon field equations remain self-dual invariant in Abelian and non-Abelian gauges.

Adopting the procedures of writing dyad components in terms of spin coefficients explicitly, we write the generalized Yang–Mills field equations of dyons in the following null tetrad notation:

$$\begin{aligned}\bar{\delta}\phi_{0k} - D\phi_{1k} = {} & (2\alpha - \pi)\phi_{0k} - 2p\phi_{1k} + k\phi_{2k} \\ & + g\epsilon_{klm}(\phi_{0l}V_{10'm} - \phi_{1l}V_{00'm}) + 2\pi J_{00'k}\end{aligned} \tag{7.45a}$$

$$\begin{aligned}\bar{\delta}\phi_{1k} - D\phi_{2k} = {} & \lambda\phi_{0k} - 2\pi\phi_{1k} + (2\epsilon - p)\phi_{2k} \\ & + g\epsilon_{klm}(\phi_{1l}V_{10'm} - \phi_{2l}V_{00'm}) + 2\pi J_{10'k}\end{aligned} \tag{7.45b}$$

$$\begin{aligned}\nabla\phi_{0k} - \delta\phi_{1k} = {} & (2T - \mu)\phi_{0k} - 2T\phi_{1k} + \sigma\phi_{2k} \\ & + g\epsilon_{klm}(\phi_{0l}V_{11'm} - \phi_{1l}V_{01'm}) + 2\pi J_{01'k}\end{aligned} \tag{7.45c}$$

$$\begin{aligned}\nabla\phi_{1k} - \delta\phi_{2k} = {} & v\phi_{0k} - 2\mu\phi_{1k} + (2\beta - T)\phi_{2k} \\ & + g\epsilon_{klm}(\phi_{1l}V_{11'm} - \phi_{2l}V_{01'm}) + 2\pi J_{11'k}\end{aligned} \tag{7.45d}$$

These equations are just the generalizations of our Maxwell–Dirac equations of dyons with the latter as described by Bisht *et al.* (1991*b*) in null tetrad notations. Equations (7.45) are obtained by doubling Carmeli's equations and defining quantities like ϕ, V, and J (as complex quantities) associated with dyons carrying electric and magnetic counterparts as their real and imaginary constituents. Let us write the line element of Minkowskian space-time $\{x^\mu\}$ in retarded (null) coordinates $x^0 = u, x^1 = r, x^2 = \theta, x^3 = \phi$ as

$$ds^2 = du^2 + 2\,du\,dr - r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{7.46}$$

where the retarded time coordinate takes the value $u = t - r$ and use of natural units $c = \hbar = 1$ is made throughout. In this null coordinate system l^μ, m^μ, and n^μ are defined as

$$\begin{aligned} l^\mu &= \delta_1^\mu \\ m^\mu &= 1/\sqrt{2}r(\delta_2^\mu + i/\sin\theta\,\delta_3^\mu) \\ n^\mu &= \delta_0^\mu - \tfrac{1}{2}\delta_1^\mu \end{aligned} \tag{7.47}$$

This choice of tetrad yields the following values of spin coefficients:

$$\begin{aligned} &\epsilon = k = \pi = \sigma = \lambda = \tau = T = v = 0 \\ &p = 2\mu = 1/r; \qquad \beta = -\alpha = 1/2\sqrt{2}r\cot\theta \end{aligned} \tag{7.48}$$

and the directional derivatives get modified into the following explicit forms:

$$D = \partial/\partial r; \qquad \delta = 1/\sqrt{2}r\mathscr{D}; \qquad \nabla = \partial/\partial u - \tfrac{1}{2}\partial/\partial r \tag{7.49}$$

where

$$\mathscr{D} = \partial/\partial\theta + i/\sin\theta\,\partial/\partial\phi$$

Substituting these expressions in equations (7.37)–(7.39), we get the following relationships between the generalized Yang–Mills potential and fields of dyons:

$$\begin{aligned} \phi_{0k} = {} & 1/\sqrt{2}rDV_{00'k} - (\partial/\partial r + 1/r)V_{01'k} \\ & + g\epsilon_{klm}V_{00'l}V_{01'm} \end{aligned} \tag{7.50}$$

$$\begin{aligned} \phi_{1k} = {} & \tfrac{1}{2}\{(\partial/\partial u - \tfrac{1}{2}\partial/\partial r)V_{00'k} + 1/\sqrt{2}r[(\mathscr{D} + \cot\theta)V_{10'k} \\ & - (\bar{\mathscr{D}} + \cot\theta)V_{10'k}] - \partial/\partial r\,V_{11'k} \\ & + g\epsilon_{mlk}(V_{00'l}V_{11'm} - V_{01'l}V_{10'm})\} \end{aligned} \tag{7.51}$$

$$\begin{aligned} \phi_{2k} = {} & (\partial/\partial u - \tfrac{1}{2}\partial/\partial r - \tfrac{1}{2}r)V_{10'k} - \tfrac{1}{2}r\bar{\mathscr{D}}V_{11'k} \\ & + g\epsilon_{mlk}V_{10'l}V_{11'm} \end{aligned} \tag{7.52}$$

Similarly, the Yang–Mills field equations given by equation (7.45) get modified to the following equations of dyons:

$$(\partial/\partial r + 2/r)\phi_{1l}$$
$$= 1/\sqrt{2}r(\bar{\mathscr{D}} + \cot\theta)\phi_{0l} + g\epsilon_{klm}(\phi_{0k}V_{10'm} - \phi_{1k}V_{00'm}) + 2\pi J_{00'l} \quad (7.53a)$$

$$(\partial/\partial u + -\tfrac{1}{2}r - \tfrac{1}{2}\,\partial/\partial r)\phi_{0l}$$
$$= \tfrac{1}{2}r\bar{\mathscr{D}}\phi_{1l} + g\epsilon_{kml}(\phi_{0k}V_{11'm} - \phi_{1k}V_{01'm}) + 2\pi J_{01'l} \quad (7.53b)$$

$$(\partial/\partial r + 1/r)\phi_{2l}$$
$$= 1/\sqrt{2}r\mathscr{D}\phi_{1l} - g\epsilon_{kml}(\phi_{1k}V_{10'm} - \phi_{2k}V_{00'm}) + 2\pi J_{10'l} \quad (7.53c)$$

$$(\partial/\partial u - \tfrac{1}{2}\,\partial/\partial r - 1/r)\phi_{1l}$$
$$= 1/\sqrt{2}r(\mathscr{D} + \cot\phi)\phi_{2l} + g\epsilon_{kml}(\phi_{1k}V_{11'm} - \phi_{2k}V_{01'm}) + 2\pi J_{11'l} \quad (7.53d)$$

In order to seek the solutions of the generalized Yang–Mills field equations of dyons, let us start with our complex field strength spinors ϕ_{10k} and ϕ_{21k} and use $\phi_{0k} = \phi_{00k}$, $\phi_{1k} = \phi_{01k} = \phi_{10k}$, and $\phi_{2k} = \phi_{11k}$ and $\sigma^{\mu}_{ab'}$ is a tetrad of null vectors with

$$\begin{aligned} l_\alpha &= \delta^0_\alpha \\ m_\alpha &= -r/\sqrt{2}\ (\delta^2_\alpha + i/\sin\theta\,\delta^3_\alpha) \\ n_\alpha &= \tfrac{1}{2}\delta^0_\alpha - \tfrac{1}{2}\delta^1_\alpha \end{aligned} \quad (7.54)$$

The functions $V_{00'k}$, $V_{01'k}$, $V_{10'k}$, $V_{11'k}$ are the potential isotriplet related to $V_{\mu k}$ by

$$V_{\mu k} = n_\mu v_{00'k} - \bar{m}_\mu V_{01'k} - m_\mu V_{10'k} + l_\mu V_{11'k} \quad (7.55)$$

and $V_{01'k} = V_{10'k}$. The angular dependence of the complex field strength spinors ϕ_0, ϕ_1, and ϕ_2 is expressed as

$$\begin{aligned} \phi_0 &\approx D^J_{1m}(\theta, \phi) \\ \phi_1 &\approx D^J_{0m}(\theta, \phi) \\ \phi_2 &\approx D^J_{-1m}(\theta, \phi) \end{aligned} \quad (7.56)$$

and

$$\begin{aligned} V_{00'} &\approx D^J_{0m}(\theta, \phi) \\ V_{01'} &\approx D^J_{1m}(\theta, \phi) \\ V_{10'} &\approx D^J_{-1m}(\theta, \phi) \\ V_{11'} &\approx D^J_{0m}(\theta, \phi) \end{aligned} \quad (7.57)$$

where $D^J_{mn}(\theta, \phi)$ are the matrix elements of a reducible representation of the group $SU(2)$ in internal isospin space where the isospin index fixes the second index of matrix elements D_{NM}. Thus

$$\begin{aligned}\phi_{0,\pm 1} &\approx D^J_{1,\pm 1}(\theta, \phi)\\ \phi_{0,3} &\approx D^J_{1,0}(\theta, \phi)\end{aligned} \tag{7.58a}$$

and

$$\phi_{0,\pm 1} = 1/\sqrt{2}i(\phi_{01} + \phi_{02}); \qquad \phi_{0,3} = \phi_{0,3} \tag{7.58b}$$

To obtain the dyon solution, we take the $J = 1$ case and the assumption of an r^{-1} dependence of potentials. Then we get

$$\begin{aligned}V_{00'k} &= 2a/grn_k(\theta, \phi); \qquad V_{01'k} = ie/\sqrt{2}gr\mathscr{D}n_k(\theta, \phi)\\ V_{10'k} &= -ie/\sqrt{2}gr\bar{\mathscr{D}}n_k(\theta, \phi); \qquad V_{11'k} = b/grn_k(\theta, \phi)\end{aligned} \tag{7.59}$$

where a, b, and e are arbitrary real constants and $n_k(\theta, \phi)$ is a unit vector given by

$$n_k(\theta, \phi) = (\sin\theta \cos\phi, \sin\theta, \sin\phi, \cos\theta) \tag{7.60}$$

along with the requirement $\phi_{0k} = \phi_{2k} = 0$ and

$$\phi_{1k} = (a + b + i)/2gr^2 n_k \tag{7.61}$$

The Yang–Mills potential and the field strength of dyons satisfy the Yang–Mills field equations without sources on making $J_{00'}$, $J_{10'}$, $J_{01'}$, and $J_{11'}$ in equation (7.53) vanish. Thus the dyonic Yang–Mills equations without sources are symmetrical and identical for electric and magnetic potentials. Two sets of Yang–Mills equations of dyons (i.e., for real and imaginary parts of dyonic potentials and field strength spinors) are the same if one imposes the duality conditions (7.41)–(7.44). In this case two potentials and field strength spinors of dyons reduce to one potential and one field strength spinor, i.e.,

$$\phi = \psi, \qquad V = p, \qquad g_{\mu\nu} = f_{\mu\nu} \tag{7.62}$$

As such one can directly write the following forms of potentials and field strengths:

$$\begin{aligned}p_{0k} &= (a + b)/g \cdot x^k/r^2\\ p_{jk} &= 1/g\{-\epsilon_{jkl}x^1/r^2 + (a + b)x^j x^k/r^3\}\\ f_{0jk} &= -(a + b)/g \cdot x^j x^k/r^4\\ f_{ijk} &= 1/g\epsilon_{ijn}x^n x^k/r^4\end{aligned} \tag{7.63}$$

where flat space-time coordinates are taken, $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$.

Similar solutions can also be written for magnetic potential and field strength if one takes $\phi \approx \psi$, $V \approx q$, and $g_{\mu\nu} \approx h_{\mu\nu}$. Now choosing $a = b$ and $a + b = e$, one gets,

$$\begin{aligned} p_{0k} &= e/g \cdot x^k/r^2 \\ p_{jk} &= -1/g \cdot \epsilon_{jkl} \cdot x^1/r^2 \\ f_{0jk} &= -e/g \cdot x^j x^k/r^4 \\ f_{ijk} &= 1/g \cdot \epsilon_{ijn} \cdot x^n x^k/r^4 \end{aligned} \tag{7.64}$$

On the other hand, for the magnetic potential, we choose $a = b$ and $a + b = q$ to maintain the duality between electric and magnetic constituents. In the former case the potential satisfies the condition in Lorentz gauge $\partial_\mu A^\mu_k = 0$, while for the latter case it satisfies $\partial_\mu B^\mu_k = 0$. In both cases the solutions reduce to the Yang–Mills solutions (Wu and Yang, 1969) of pure monopole (electric charge) as well as the Julia–Zee (1975) solutions of dyons. The constant e may be interpreted as g times the electric charge of a dyon. Here g is the Yang–Mills coupling constant and is different from the magnetic charge of Section 2. These solutions represent the field of dyons which has both an electric (magnetic) charge e/g and a magnetic (electric) charge $1/g$. Maxwell's field strength spinor for dyons then reduces to

$$\phi_0 = \phi_2 = 0 \qquad \text{and} \qquad \phi_1 = \frac{e + ig}{2r^2} \tag{7.65}$$

For these solutions the Wu–Yang (1969) solutions are found by taking electric charge e vanishing, while the Julia–Zee (1975) dyon solutions can be written by modifying equation (7.64) as follows

$$A^a_0 = p^a_0 = x^a/r \cdot J(r)/er \tag{7.66}$$

$$A^a_i = p^a_i = \epsilon_{abi} x^b/r^2[k(r) - 1]/er \tag{7.67}$$

The foregoing analysis shows that the null tetrad method of Yang–Mills field equations can be extended well for particles carrying electric and magnetic charges (dyons). To seek the solutions of source-free Yang–Mills field equations one can impose the duality conditions between electric and magnetic constituents. Our solutions and field equations of dyons resemble the investigations of Carmeli (1977*a*, 1982), with the difference that we have formulated our theory from the beginning by means of two potentials, while the later described the same in terms of one potential. Second, our theory has two types of solutions in terms of two Yang–Mills gauge potentials. The two-potential approach is necessary to remove the arbitrary string variable in Dirac's (1931) theory of the magnetic monopole. Consequently the Maxwell equations without sources are symmetrical. We do not find any use of two potentials in order to seek the solutions of source-free Yang–Mills equations. Still one is free to describe any one potential for seeking the dyon solutions. As such, two-potential theory is the consequence of the Abelian gauge theory of dyons, while one Yang–Mills potential is sufficient in non-Abelian gauge theory where monopoles appear as classical solutions of Yang–Mills source-free equations. The two potential approach describes well the duality between electric and magnetic constituents even in Yang–Mills gauge theory and the solutions of Yang–Mills equations having both electric and magnetic charges.

ACKNOWLEDGMENTS

One of us (O.P.S.N.) is thankful to Prof. H. Dehnen for his warm hospitality at the Universität Konstanz. Financial assistance from the Council of Scientific and Industrial Research, New Delhi, to P.S.B. and from German Academic Exchange Services (DAAD) Bonn to O.P.S.N. is also acknowledged.

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